The Ricci Flow on compact Kahler manifold

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(X, g_0) closed Riemannian manifold (without boundary)

$$\begin{cases} \frac{\partial g(t)}{\partial t} = -Rc(g(t)) \\ g(0) = g_0 \end{cases}$$ Rc: Ricci flow

each $g(t)$ is a R-structure

If $\dim X \geq 4$, Ricci flow is poorly understood.

Restrict on Kahler compact manifold.

- X compact complex manifold
- Multiplication by $i \Rightarrow J: TX \to TX$ $J^2 = -Id$
- g (R-structure) is Hermittian if $g(TX, TY) = g(X, Y)$

Then $\omega(X, Y) := g(JX, Y) \leftarrow$ 2-form called real (1,1)-form

<u>Def</u> g is Kähler if $d\omega = 0$

- Every Riemannian Surface ($\dim_{\mathbb{C}} = n = 1$) automatically Kähler
- tori $\mathbb{C}^n/\Lambda$ $\Lambda = \mathbb{Z}^{2n}$
- $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$ g = Fubinis Study Metric
- complex submanifold of Kähler are Kähler
- all $X \hookrightarrow \mathbb{CP}^N$ are Kähler

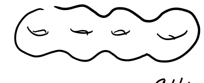
(Kodaira: X compact complex is proj $\Leftrightarrow \exists g$ Kahler X s.t. $[\omega] \in H^2(X, \mathbb{Q})$)

Hamilton: (X, g_0) cpt Kähler, $\begin{cases} \frac{\partial g}{\partial t} = -Rc(g(t)) \\ g(0) = g_0 \end{cases}$ has a solution $g(t)$

defined on a maximal interval $t \in [0, T)$, $0 < T \leq \infty$

and $g(t)$ are also Kahler.

What happens at $t \to T$?

$(n=1)$ Riemann Surfaces X is diffeo to Σ_g  $\leftarrow$ genus g

<u>Thm</u> (Hamilton, Chow) · if $g \geq 2$, then $T = \infty$ and $\frac{g(t)}{t} \xrightarrow{C^\infty} g_{hyp}$, $Rc(g_{hyp}) = -g_{hyp}$

- if $g = 1$, then $T = \infty$, $g(t) \xrightarrow{C^\infty} g_{flat}$, $Rc(g_{flat}) = 0$
- if $g = 0$, then $T < \infty$ and $\frac{g(t)}{T-t} \xrightarrow{C^\infty} g_{round}$, $Rc(g_{round}) = g_{round}$

How about general dimensions?

(X, g) cpt Kähler $\rightsquigarrow \omega$ 2-form $\rightsquigarrow [\omega] \in H^{1,1}(X, \mathbb{R}) \subset H^2(X, \mathbb{R})$

$Rc(X, Y) = Rc(JX, JY)$ $\rightsquigarrow$ then $Ric(X, Y) = Rc(JX, Y)$ closed $(1,1)$ form

$[Ric(g)] =: 2\pi c_1(X)$ first Chern class, indep of g

KRF $\iff \begin{cases} \frac{\partial g(t)}{\partial t} = -Ric(\omega(t)) \\ \omega(0) = \omega_0 \end{cases}$

$\Rightarrow$ set $[\omega(t)] = [\omega_0] - 2\pi t c_1(X)$ for $t < T$

$\mathcal{C}_X = \{ \alpha \in H^{1,1}(X, \mathbb{R}) \mid \exists \omega$ Kahler form on X with $[\omega] = \alpha \}$

Kahler cone of X

Thm [Tsuji, Tian-Zhang] (X, g_0) cpt Kahler, Then

$T = \sup \{ t > 0 \mid [\omega_0] - 2\pi t c_1(X) \in \mathcal{C}_X \}$

Hence $T = \infty \iff -c_1(X) \in \overline{\mathcal{C}_X}$ "K_X is nef" "X is a minimal model"

Case $T < \infty$ a singularity forms Call $\alpha_i = [\omega] - 2\pi T c_1(X) \in \partial \mathcal{C}_X$

Conj (Feldman - Ilmanen - Knopf, 03) in this case, the singularities formation set is an analytic subvariety of X

then say form set Σ, $\Sigma^c := \{ x \in X \mid \exists U$ open of x, $\exists \omega_T$ Kahler over U s.t. $\omega(t) \xrightarrow[t \to T]{C^\infty(U)} \omega_T \}$

Thm (T. Collins - T.B) conj is true. $\Sigma = Null(\alpha)$ null locus of α

where $Null(\alpha) = \bigcup_{V \subset X \text{ irred analytic subvar}} V$